

On some \mathcal{D} -modules in dimension 2

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Abstract.- We prove a duality formula for two \mathcal{D} -modules arising from logarithmic derivations w.r.t. a plane curve. As an application we give a differential proof of a logarithmic comparison theorem in [4].

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1 Introduction

Let $\mathcal{O} = \mathbf{C}\{x, y\}$ be the ring of convergent power series in two variables and \mathcal{D} the ring of linear differential operators with coefficients in \mathcal{O} . For each reduced power series $f \in \mathcal{O}$, with $f(0, 0) = 0$, we will denote by I^{log} the left ideal of \mathcal{D} generated by the logarithmic derivations (see [11]) with respect to f . We denote by $\text{Der}_{\mathbf{C}}(\mathcal{O})$ the Lie algebra of \mathbf{C} -derivations on \mathcal{O} . Recall that a derivation $\delta \in \text{Der}_{\mathbf{C}}(\mathcal{O})$ is logarithmic if there exists $a \in \mathcal{O}$ such that $\delta(f) = af$. We denote by \tilde{I}^{log} the left ideal of \mathcal{D} generated by the operators of the form $\delta + a$ where $\delta(f) = af$.

We first prove that the \mathcal{D} -modules $M^{log} = \mathcal{D}/I^{log}$ and $\tilde{M}^{log} = \mathcal{D}/\tilde{I}^{log}$ are dual each to the other and then that both \mathcal{D} -modules are regular holonomic 2.3.

Let $\mathcal{O}[1/f]$ be the \mathcal{D} -module of (the germs of) the meromorphic functions in two variables with poles along f . There exists a natural surjective

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morphism $\psi : \widetilde{M}^{log} \rightarrow \mathcal{O}[1/f]$. Using [4] we prove that $\text{Ext}_{\mathcal{D}}^2(\widetilde{M}^{log}, \mathcal{O}) = 0$ if and only if f is quasi-homogeneous and then we obtain that the morphism ψ is an isomorphism if and only if f is quasi-homogeneous (see 2.5). As a consequence we give a new “differential” proof of the logarithmic comparison theorem of [4].

These results are susceptible to be generalized to the case of higher dimensions but no general results are known up to now. See [6] for a proof of the duality formula in higher dimension. Nevertheless we give a complete example showing that some results of the present work are true in dimension 3.

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2 The module \widetilde{M}^{log} in the general case.

Let us consider any reduced $f \in \mathcal{O} = \mathbf{C}\{x, y\}$ with a singular point at the origin. It is possible to obtain, from the logarithmic derivations, an ideal inside $\text{Ann}_{\mathcal{D}}(1/f)$: if $\delta(f) = af$ then $\delta + a \in \text{Ann}_{\mathcal{D}}(1/f)$. This fact suggested us a general way to present the annihilating ideal of $1/f$ for a constructive proof of the equality $\text{Ext}_{\mathcal{D}}^2(\mathcal{O}[1/f], \mathcal{O}) = 0$ for any “polynomial” curve (see [14]).

We have $\widetilde{I}^{log} \subset \text{Ann}_{\mathcal{D}}(1/f)$, where \widetilde{I}^{log} is the left ideal in \mathcal{D} generated by the operators $\delta + a$ for $\delta \in \text{Der}_{\mathbf{C}}(\mathcal{O})$ and $\delta(f) = af$. Then we have a surjective morphism $\psi : \widetilde{M}^{log} = \frac{\mathcal{D}}{\widetilde{I}^{log}} \longrightarrow \mathcal{D}/\text{Ann}_{\mathcal{D}}(1/f) \simeq \mathcal{O}[1/f]$ (for the last isomorphism we use that the Bernstein polynomial of f has no integer roots smaller than -1 (see [15])). It is well known that around each smooth point of $f = 0$ the morphism ψ is in fact an isomorphism. So, the kernel K of ψ is a \mathcal{D} -module concentrated at the origin. Then K is a direct sum of “couches-multiples” modules [8], and this type of modules are regular holonomic [9]. In particular \widetilde{M}^{log} is regular holonomic because $\mathcal{O}[\frac{1}{f}]$ and K are.

We will denote by $\text{Der}(\log f)$ the Lie algebra of logarithmic derivations with respect to f . By [11] $\text{Der}(\log f)$ is a free \mathcal{O} -module of rank two. Let $\{\delta_1, \delta_2\}$ be a basis of $\text{Der}(\log f)$,

$$\begin{cases} \delta_1 &= b_1 \partial_x + c_1 \partial_y, \\ \delta_2 &= b_2 \partial_x + c_2 \partial_y \end{cases}.$$

We can suppose that

$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = f$$

We will take into account the following results for any (reduced) curve f :

- Every basis δ_1, δ_2 of $\text{Der}(\log f)$ verifies that

$$\langle \sigma(\delta_1), \sigma(\delta_2) \rangle = \text{gr}^F(I^{\log}) = \text{gr}^F(\tilde{I}^{\log}),$$

because $\{\sigma(\delta_1), \sigma(\delta_2)\}$ is a regular sequence (see [2] and ([3], Corollary 4.2.2)). Here $\sigma(\cdot)$ denotes the principal symbol of the corresponding operator and $\text{gr}^F(I^{\log})$ is the graded ideal associated to the order filtration on \mathcal{D} . Therefore,

$$CCh(\tilde{M}^{\log}) = CCh(M^{\log}),$$

where $CCh(\cdot)$ represents the *characteristic cycle* of the \mathcal{D} -module (see, for example, [7]). Of course both M^{\log} and \tilde{M}^{\log} define coherent \mathcal{D} -modules in some neighborhood of the origin and then we can properly speak of characteristic varieties and characteristic cycles. Since \tilde{M}^{\log} is holonomic then M^{\log} is holonomic.

- For any curve,

$$\text{Sol}(M^{\log}) \xrightarrow{q.i.} \Omega^\bullet(\log f) \xrightarrow{\varphi} \Omega^\bullet[1/f] \simeq DR(\mathcal{O}[1/f]),$$

where $\text{Sol}(\cdot)$ and $DR(\cdot)$ are the *solutions complex* and the *De Rham complex* (see, for example, [9]) and where $\Omega^\bullet(\log f)$ (resp. $\Omega^\bullet([1/f])$) is the complex of logarithmic differential forms (resp. meromorphic differential forms). The first quasi-isomorphism appears in [3] and φ is the natural morphism.

Proposition 2.1 *Let f be a (reduced) curve and let $\{\delta_1, \delta_2\}$ be a basis of $\text{Der}(\log f)$ with $[\delta_1, \delta_2] = \alpha_1\delta_1 + \alpha_2\delta_2$ and $\delta_i(f) = a_i f$, $i = 1, 2$. Then*

$$\mathcal{D}\{\delta_2^t + \alpha_1, \delta_1^t - \alpha_2\} = \mathcal{D}\{\delta_1 + a_1, \delta_2 + a_2\}$$

where δ_i^t is the transposed of δ_i .

Proof: First we find an expression of the α_i from the a_j, b_k, c_l :

$$\begin{aligned}
[\delta_1, \delta_2] &= \alpha_1(b_1\partial_x + c_1\partial_y) + \alpha_2(b_2\partial_x + c_2\partial_y) = \\
&= (\alpha_1b_1 + \alpha_2b_2)\partial_x + (\alpha_1c_1 + \alpha_2c_2)\partial_y = \\
&= b_1\partial_x(b_2)\partial_x - b_2\partial_x(b_1)\partial_x + b_1\partial_x(c_2)\partial_y - c_2\partial_y(b_1)\partial_x + \\
&+ c_1\partial_y(b_2)\partial_x - b_2\partial_x(c_1)\partial_y + c_1\partial_y(c_2)\partial_y - c_2\partial_y(c_1)\partial_y = \\
&= (c_1\partial_y(b_2) - b_2\partial_x(b_1) - c_2\partial_y(b_1) + c_1\partial_y(b_2))\partial_x + \\
&+ (b_1\partial_x(c_2) - b_2\partial_x(c_1) - c_2\partial_y(c_1) + c_1\partial_y(c_2))\partial_y.
\end{aligned}$$

Besides,

$$-\delta_1^t + \alpha_2 = \partial_x b_1 + \partial_y c_1 + \alpha_2 = \delta_1 + \alpha_2 + \partial_x(b_1) + \partial_y(c_1).$$

To prove that $\alpha_2 + \partial_x(b_1) + \partial_y(c_1) = a_1$, we will establish that

$$\alpha_2 f = a_1 f - \partial_x(b_1)f - \partial_y(c_1)f$$

We have

$$\begin{aligned}
&a_1 f - \partial_x(b_1)f - \partial_y(c_1)f = \\
&= (b_1\partial_x + c_1\partial_y) - \partial_x(b_1) - \partial_y(c_1))(b_1c_2 - b_2c_1) = \\
&= b_1(b_1\partial_x(c_2) - c_1\partial_x(b_2) - b_2\partial_x(c_1)) + \\
&+ c_1(c_2\partial_y(b_1) + b_1\partial_y(c_2) - c_1\partial_y(b_2)) + \\
&+ b_2c_1\partial_x(b_1) - b_1c_2\partial_y(c_1).
\end{aligned}$$

Therefore

$$(\alpha_1, \alpha_2) \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} = (\gamma_1, \gamma_2),$$

where

$$\begin{aligned}
\gamma_1 &= c_1\partial_y(b_2) - b_2\partial_x(b_1) - c_2\partial_y(b_1) + c_1\partial_y(b_2), \\
\gamma_2 &= b_1\partial_x(c_2) - b_2\partial_x(c_1) - c_2\partial_y(c_1) + c_1\partial_y(c_2).
\end{aligned}$$

Multiplying by the transposed adjoint matrix and by f we obtain

$$(\alpha_1 f, \alpha_2 f) = (\gamma_1, \gamma_2) \begin{pmatrix} c_2 & -c_1 \\ -b_2 & b_1 \end{pmatrix}.$$

and hence the equality follows. In a similar way $\delta_2^t + \alpha_1 = -\delta_2 - a_2$. Then both ideals are equal. \square

Prof. Narváez pointed us to consider, instead of the Lie algebra $\text{Der}(\log f)$, the Lie algebra

$$L = \{\delta + a \mid \delta(f) = af\},$$

and try to construct of a free resolution (of “Spencer type”) of \widetilde{M}^{\log} ([2], [3]). In fact, we have

Proposition 2.2 *A free resolution of \widetilde{M}^{\log} is*

$$0 \longrightarrow \mathcal{D} \xrightarrow{\phi_2} \mathcal{D}^2 \xrightarrow{\phi_1} \mathcal{D} \longrightarrow \widetilde{M}^{\log} \longrightarrow 0,$$

where ϕ_2 is defined by the matrix

$$(-\delta_2 - a_2 - \alpha_1, \delta_1 + a_1 - \alpha_2),$$

and ϕ_1 by $\begin{pmatrix} \delta_1 + a_1 \\ \delta_2 + a_2 \end{pmatrix}$.

Proof: To check the exactness of the resolution above, it is enough to consider a discrete filtration on that complex and to verify the exactness of the resulting resolution (see [1], chapter 2, lemma 3.13). The same argument is used in [2], [3](proposition 4.1.3) to prove that the complex $\mathcal{D} \otimes_{\mathcal{V}_0^f(\mathcal{D})} Sp^\bullet(\log f)$ is a free resolution of M^{\log} (as a left \mathcal{D} -module)¹. But, for $n = 2$, the exact graded complex in the proof of [3] is precisely

$$0 \longrightarrow \text{gr}^F(\mathcal{D}) \xrightarrow{M_1} \text{gr}^F(\mathcal{D})^2[-2] \xrightarrow{M_2} \text{gr}^F(\mathcal{D})[-1] \longrightarrow \text{gr}^F(M^{\log}) \longrightarrow 0,$$

where the matrices are

$$M_1 = (-\sigma^F(\delta_2), \sigma^F(\delta_1)), \quad M_2 = \begin{pmatrix} \sigma^F(\delta_1) \\ \sigma^F(\delta_2) \end{pmatrix}.$$

And the last complex is the result of applying the same graduation to the resolution of \widetilde{M}^{\log} too, because

$$\sigma^F(\delta_i) = \sigma^F(\delta_i + a_i).$$

□

¹Here $Sp^\bullet(\log f)$ is the Logarithmic Spencer complex and $\mathcal{V}_0^f(\mathcal{D})$ is the ring of degree zero differential operators w.r.t. \mathcal{V} -filtration relative to f . See [2], [3], section 1.2.

Proposition 2.3 *Given $f \in \mathbf{C}\{x, y\}$, $\widetilde{M}^{log} \simeq (M^{log})^*$ where $()^*$ is the dual in the sense of \mathcal{D} -modules. In particular \widetilde{M}^{log} and M^{log} are regular \mathcal{D} -modules.*

Proof: We take the free resolution of M^{log} (see [2], ([3, Th. 3.1.2]))

$$0 \longrightarrow \mathcal{D} \xrightarrow{\psi_2} \mathcal{D}^2 \xrightarrow{\psi_1} \mathcal{D} \longrightarrow M^{log} \longrightarrow 0,$$

where $\{\delta_1, \delta_2\}$ is a basis of the \mathcal{O} -module $\text{Der}(\log f)$, where

$$[\delta_1, \delta_2] = \alpha_1 \delta_1 + \alpha_2 \delta_2,$$

$$\psi_1 = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

and, on the other hand, ψ_2 is the syzygy matrix

$$\psi_2 = (-\delta_2 - \alpha_1, \delta_1 - \alpha_2).$$

Applying the $\text{Hom}_{\mathcal{D}}(-, \mathcal{D})$ functor to calculate the dual module, we obtain the sequence

$$0 \longrightarrow \mathcal{D} \xrightarrow{\psi_1^*} \mathcal{D}^2 \xrightarrow{\psi_2^*} \mathcal{D} \longrightarrow 0,$$

where ψ_2^* is the right product by $\begin{pmatrix} -\delta_2 - \alpha_1 \\ \delta_1 - \alpha_2 \end{pmatrix}$. Hence, $(M^{log})^*$ is the left \mathcal{D} -module associated to the right \mathcal{D} -module $\mathcal{D}/(\delta_2 + \alpha_1, \delta_1 - \alpha_2)\mathcal{D}$, that is to say,

$$(M^{log})^* \simeq \mathcal{D}/\mathcal{D}(\delta_2^t + \alpha_1, \delta_1^t - \alpha_2).$$

Using the proposition 2.1, we deduce that $(M^{log})^* \simeq \widetilde{M}^{log}$. The regularity of M^{log} follows from the regularity of \widetilde{M}^{log} (c.f. [9]). \square

Proposition 2.4 *If f is a non quasi homogeneous (reduced) curve, then*

$$\text{Ext}_{\mathcal{D}}^2(\widetilde{M}^{log}, \mathcal{O}) \neq 0.$$

Proof: The proof of this result contains, as an essential ingredient, a re-reading of the demonstration of Theorem 3.7 of [4]. As a matter of fact, we include some tricks of this demonstration.

By proposition 2.2, a free resolution of \widetilde{M}^{log} is

$$0 \longrightarrow \mathcal{D} \xrightarrow{\phi_2} \mathcal{D}^2 \xrightarrow{\phi_1} \mathcal{D} \longrightarrow \widetilde{M}^{log} \longrightarrow 0,$$

where ϕ_2 is the matrix

$$(-\delta_2 - a_2 - \alpha_1, \delta_1 + a_1 - \alpha_2).$$

Hence, $\text{Ext}_{\mathcal{D}}^2(\widetilde{M}^{\log}, \mathcal{O}) \simeq \mathcal{O}/\text{Img}\phi_2^*$. To guarantee that this vector space has dimension greater than zero, it is enough to show that a pair of functions $h_1, h_2 \in \mathcal{O}$ such that

$$(-\delta_2 - a_2 - \alpha_1, \delta_1 + a_1 - \alpha_2) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 1,$$

does not exist, that is to say, that $1 \notin \text{Img}\phi_2^*$.

Let us take $\delta_1 = b_1\partial_x + c_1\partial_y$. As $a_1 - \alpha_2 = \partial_x(b_1) + \partial_y(c_1)$, (proposition 2.1) we will prove that, or b_1 and c_1 have no lineal parts, or that after derivation those lineal parts become 0.

Of course f has no quadratic part: in that case, because of the classification of the singularities in two variables, f would be equivalent to a quasi homogeneous curve $x^2 + y^{k+1}$, for some k . Then we can suppose that

$$f = f_n + f_{n+1} + \cdots = \sum_{k \geq n} h_k = \sum_{k \geq n} \sum_{i+j=k} a_{ij} x^i y^j,$$

where $n \geq 3$ and $f_n \neq 0$.

We will write

$$\delta_1 = b_1\partial_x + c_1\partial_y = \delta_0^1 + \delta_1^1 + \cdots = \sum_{k \geq 0} \sum_{i+j=k+1} (\beta_{ij}^1 x^i y^j \partial_x + \gamma_{ij}^1 x^i y^j \partial_y),$$

where the linear part δ_0^1 is $(xy)A_0(\partial_x\partial_y)^t$, and A_0 is a matrix 2×2 with complex coefficients.

If $A_0 = 0$, we have finished. Otherwise, the possibilities of the Jordan form of A_0 are

$$A_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}.$$

As δ_1 is not an Euler vector (because f is not quasi homogeneous), we deduce:

- If we take the first Jordan form, then (see the cited demonstration of [4]) $f_n = x^p y^q$ y $\delta_0 = qx\partial_x - py\partial_y$. After a sequence of changes of coordinates we have that $f = x^p y^q$ with $p+q = n \geq 3$, that contradicts that f is reduced.

- For the second Jordan form with $\lambda_1 \neq 0$, it has to be $f_n = 0$, that contradicts that f has its initial part of grade n .
- For the second option with $\lambda_1 = 0$ we have $\delta_0^1 = y\partial_x$ and, in this situation, the linear of b_1 is y . If we precisely apply ∂_x , we obtain 0.

In a similar way, you prove the same for $a_2 + \alpha_1$. \square

Theorem 2.5 *The natural morphism $\widetilde{M}^{log} \xrightarrow{\psi} \mathcal{O}[\frac{1}{f}]$ is an isomorphism if and only if f is a quasi homogeneous (reduced) curve.*

Proof: As we pointed, if f is quasi homogeneous then $\widetilde{I}^{log} = Ann_{\mathcal{D}}(1/f)$ and therefore ψ is an isomorphism. Reciprocally, if ψ is an isomorphism, then $Ext_{\mathcal{D}}^2(\mathcal{O}[1/f], \mathcal{O}) \simeq Ext_{\mathcal{D}}^2(\widetilde{M}^{log}, \mathcal{O})$. Because of a result of [9], we have $Ext_{\mathcal{D}}^2(\mathcal{O}[1/f], \mathcal{O}) = 0$ and, if we take into account proposition 2.4, we obtain that f has to be quasi homogeneous. \square

Remark.- The following result can be obtain using [13]: if f is not quasi-homogeneous curve then $Ann_{\mathcal{D}}(1/f)$ could not be generated by elements of degree one in ∂ and then $Ann_{\mathcal{D}}(1/f) \neq \widetilde{I}^{log}$.

Let us give a new “differential” proof of a version of the *Logarithmic Comparison Theorem* [4].

Theorem 2.6 *The complexes $\Omega^{\bullet}(log f)$ and $\Omega^{\bullet}[1/f]$ are isomorphic in the correspondent derived category if and only if f is quasi homogeneous.*

Proof: If f is quasi homogeneous we have pointed yet that \widetilde{M}^{log} is isomorphic to $\mathcal{O}[\frac{1}{f}]$. By the proposition 2.3 $(M^{log})^{\star} \simeq \widetilde{M}^{log}$ and then we have

$$\Omega^{\bullet}(log f) \simeq Sol(M^{log}) \simeq DR((M^{log})^{\star}) \simeq DR(\widetilde{M}^{log}) \simeq \Omega^{\bullet}[1/f],$$

where the first isomorphism is obtained in [2] (see also [3]) and the second one could be found in [9]. Reciprocally, if f is not quasi homogeneous then

$\widetilde{M}^{log} \not\cong \mathcal{O}[1/f]$ and, as both are regular holonomic, neither their De Rham complexes are isomorphic, that is

$$DR(\widetilde{M}^{log}) \not\cong \Omega^\bullet[1/f],$$

using the Riemann-Hilbert correspondence of Mebkhout-Kashiwara. \square

3 Example in a constructive way of logarithmic comparison in surfaces.

We illustrate in this section an interesting example in dimension 3 of the situation presented in theorem 2.6 for curves. We consider the surface (see [2]) $h = 0$ with

$$h = xy(x + y)(xz + y),$$

which is not locally quasi-homogeneous. We prove that:

- $Ann_{\mathcal{D}}(1/h) = \widetilde{I}^{log}$.
- $\widetilde{M}^{log} \simeq (M^{log})^\star$

and we conclude the logarithmic comparison theorem holds in this case. Although this example appears in [4], here the treatment is under an effective point of view.

We can compute a basis of $\text{Der}(\log h)$ with a set of generators of the syzygies among $h, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}$. We obtain

- $\delta_1 = x\partial_x + y\partial_y,$
- $\delta_2 = xz\partial_z + y\partial_z,$
- $\delta_3 = x^2\partial_x - y^2\partial_y - xz\partial_z - yz\partial_z,$

with

$$\delta_1(h) = 4h, \quad \delta_2(h) = xh, \quad \delta_3(h) = (2x - 3y)h,$$

and

$$\begin{vmatrix} x & y & 0 \\ 0 & 0 & xz + y \\ x^2 & -y^2 & -xz - yz \end{vmatrix} = h.$$

As a multiple of the b -function of h in \mathcal{D} is

$$b(s) = (4s + 5)(2s + 1)(4s + 3)(s + 1)^3,$$

and this polynomial has no integer roots smaller than -1 , we can assure that

$$\mathcal{O}[*h] \simeq \mathcal{D} \frac{1}{h}.$$

It is easy to check that $\text{Ann}_{\mathcal{D}}(1/h)$ is equal to \tilde{I}^{log} . The computations of the b -function and the annihilating ideal of h^s have been made using the algorithms of [10], implemented in [12].

We calculate (using Gröbner bases) a free resolution of the module \mathcal{D}/I^{log} where $I^{log} = (\delta_1, \delta_2, \delta_3)$ (see [5]). The first module of syzygies is generated (in this case) by the relations deduced from the expressions of the $[\delta_i, \delta_j]$ with $i \neq j$:

- $[\delta_1, \delta_2] = \delta_2,$
- $[\delta_1, \delta_3] = \delta_3,$
- $[\delta_2, \delta_3] = -x\delta_2.$

The second module of syzygies is generated by only one element $\mathbf{s} = (s_1, s_2, s_3)$:

- $s_1 = -y^2\partial_y + x^2\partial_x - zy\partial_z - zx\partial_z - x,$
- $s_2 = -y\partial_z - xz\partial_z,$
- $s_3 = y\partial_y + x\partial_x - 2.$

The above calculations provide a free resolution of M^{log} . With a procedure similar to the used in 2.3 we obtain that $(M^{log})^*$ is the left \mathcal{D} -module associated to the right \mathcal{D} -module $\mathcal{D}/(s_1, s_2, s_3)\mathcal{D}$. Then

$$(M^{log})^* \simeq \mathcal{D}/(s_1^t, s_2^t, s_3^t).$$

It is enough to compute s_1^t, s_2^t, s_3^t and check (using Gröbner basis) that they span \tilde{I}^{log} . Hence

$$(M^{log})^* = (\mathcal{D}/\text{Der}(\log(h)))^* \simeq \mathcal{D}/\tilde{I}^{log} = \tilde{M}^{log}.$$

At this point we have obtained that

$$Sol(M^{log}) \simeq DR((M^{log})^\star) \simeq DR(\widetilde{M}^{log}) \simeq \Omega^\bullet[1/h]$$

where the last two isomorphism are due to our computations (the first was used in the proof of 2.6). Taking into account that $\Omega^\bullet(\log h) \simeq Sol(M^{log})$ (it was showed for this example in [2],[3]) we can deduce that the logarithmic comparison theorem (i.e. $\Omega^\bullet(\log h) \simeq \Omega^\bullet[1/h]$, [4]) holds without the “locally quasi homogeneous” hypothesis. It is interesting to remark too (see [2]) that $\{\sigma^F(\delta_1), \sigma^F(\delta_2), \sigma^F(\delta_3)\}$ do not form a regular sequence in $\text{gr}^F(\mathcal{D})$. We have $z\eta\zeta - \xi\zeta \notin \langle \sigma^F(\delta_1), \sigma^F(\delta_2) \rangle$ and

$$(z\eta\zeta - \xi\zeta)\sigma^F(\delta_3) \in \langle \sigma^F(\delta_1), \sigma^F(\delta_2) \rangle.$$

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